THE CODIMENSIONS OF A *PI*-ALGEBRA

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ABSTRACT

We simplify the numerical calculations given in a previous paper by Regev and obtain a much better estimation for the sequence of codimensions of a *PI*- algebra.

1. Introduction

It was proved in [1] that the tensor product of two *PI*-algebras (over a commutative ring with 1) is a *PI*-algebra. The proof was based on some bound for the rate of growth of the sequence of codimensions of a *PI*-algebra. To state the result of the present note, we introduce the following notations. Let $\mathscr{A}(d)$ be the class of *PI*-algebras satisfying an identity of degree d. For a *PI*-algebra R, denote its sequence of codimensions by $\{c_n(R)\}$. In addition we make the following

DEFINITION. Let $d > 0$ be an integer, then

$$\alpha(d) = \inf\{\beta \mid c_n(R) \leqq \beta^n, \text{ for all } R \in \mathscr{A}(d) \text{ and } n \geqq 1\}.$$

Now, the result of [1, Th. 4.7] is: $\alpha(d) \leqq 3.4^{d-3} = O(4^d)$. The aim of this note is to simplify and improve the numerical calculations given in [1, Section 4]. We shall obtain a much better estimation for $\alpha(d)$, namely, $\alpha(d) < 3(d^2 - 7d + 16) = O(d^2)$. Finally we give examples of *PI*-algebras whose sequences of codimensions can be computed.

2. A short proof of Th. 4.6

We shall assume that the reader is well acquainted with the notations and results of [1, Sections 1–3] and we shall refer frequently to this paper.

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For $d \geqq 4$, we introduce the following notations:
$L = (l_1, \cdots, l_{d-3})$, $l = \sum_{\mu=1}^{d-3} l_\mu$ and $E_1, \cdots, E_{d-3}$ the standard unit vectors with $d-3$ coordinates. Using these notations, the result of Th. 3.6. of [1] can be written as follows:

$$a(L,n) \tag{1}$$
$$\leqq \sum_{\nu=1}^{d-3} \sum_{t=l_{\nu-1}+1}^{l_\nu} a(L+(t-1-l_\nu)E_\nu, n-1) + \sum_{t=l_{d-3}+1}^{n-2} a(L,t) + 2a(L,n-1).$$

We now prove by induction on n that

$$a(L,n) \leqq 3^n 2^{n\ (d-3)+l} \tag{2}$$

and the result of Th. 4.6 will follow since

$$c_n = a(nE_1 + \cdots + nE_{d-3}, n) \leqq 3^n 2^{n(d-3)+n(d-3)} = (3.4^{d-3})^n.$$

For $n = 1$, (2) holds trivially, so let us assume that $a(L,t) \leqq 3^t 2^{t(d-3)+l}$ for $t \leqq n-1$ and $l_1 \leqq \cdots \leqq l_{d-3} \leqq t$. By [1, Prop. 1.9], $a(L,n) = a(l_1, \cdots, l_{d-3}, n) = a(h_1, \cdots, h_{d-3}, n)$, where $h_1 \leqq \cdots \leqq h_{d-3} \leqq n-2$ and $h_\nu \leqq l_\nu$ for all ν, so we may assume that $l_{d-3} \leqq n-2$. By the induction hypothesis

$$a(L+(t-1-l_\nu)E_\nu, n-1) \leqq 3^{n-1} 2^{(n-1)(d-3)+l+t-1-l_\nu}$$

for $l_{\nu-1}+1 \leqq t \leqq l_\nu$, and since

$$\sum_{t=l\ \ _{-1}+1}^{l_\nu} 2^{t-1} < 2^{l_\nu}$$

it follows that the first term of (1) is $\leqq (d-3)3^{n-1}2^{(n-1)(d-3)+l}$. For the second term of (1), we obtain similarly that it is $\leqq 3^{n-1}2^{(n-1)(d-3)+l}$. It follows that $a(L,n) \leqq (d-3+1+2)3^{n-1}2^{(n-1)(d-3)+l} = d3^{n-1}2^{(n-1)(d-3)+l}$. Since $d \leqq 3.2^{d-3}$ for $d \geqq 3$, we obtain that $a(L,n) \leqq 3^n 2^{n(d-3)+l}$.

3. A new estimation for c_n

We now dispense with simplicity. Using the above method, we achieve a much better estimation for $\alpha(d)$ and hence for the codimensions.

Define $A(L,n)$ as in [1]; then by Prop. 4.2, $a(L,n) \leqq A(L,n)$. Let $A(n) = A(nE_1 + \cdots + nE_{d-3}, n)$; then $c_n(R) \leqq A(n)$ for any algebra $R \in \mathscr{A}(d)$. Hence, if we show that $A(n) \leqq \beta^n$, then $\alpha(d) \leqq \beta$.

For $d = 3$, (1) reduces to

$$a(n) \leqq \sum_{t=1}^{n-2} a(t) + 2a(n-1), \text{ so } A(n) = \sum_{t=1}^{n-2} A(t) + 2A(n-1).$$

From this we get the difference equation $A(n+1) = 3A(n) - A(n-1)$ and $A(1) = 1$, $A(2) = 2$. The solution is given by

$$A(n) = \frac{\sqrt{5}-1}{2\sqrt{5}} \left(\frac{3+\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}+1}{2\sqrt{5}} \left(\frac{3-\sqrt{5}}{2}\right)^n.$$

So we have $\alpha(3) \leqq \frac{1}{2}(3+\sqrt{5})$ or $\sup_n \sqrt[n]{A(n)} = \frac{1}{2}(3+\sqrt{5})$, i.e., $\frac{1}{2}(3+\sqrt{5})$ is the best estimation issued from (1) for $\alpha(3)$.

From now let $d \geqq 4$. First we prove the following

LEMMA. *Let $x, y > 1$ be such that*

$$\frac{d-3}{y-1} + \frac{1}{x-1} + 2 = x. \tag{3}$$

Then

$$A(L,n) \leqq x^n y^l.$$

PROOF. As in the previous section, we prove the result by induction on n. The result is trivial for $n = 1$ and the induction hypothesis is $A(L,t) \leqq x^t y^l$. We have

$$A(L + (t-1-l_\nu)E_\nu, n-1) \leqq x^{n-1} y^{l+t-1-l_\nu}. \quad \text{Since} \quad y > 1.$$

$$\sum_{t=l_{\nu-1}+1}^{l_\nu} y^{-1-l_\nu} < \frac{1}{y-1}$$

and this implies that the first term of (1) is $\leqq \dfrac{1}{x-1}\, x^{n-1} y^l$, so by (3) it follows that:

$$A(L,n) \leqq \frac{d-3}{y-1}\, x^{n-1} y^l + \frac{1}{x-1}\, x^{n-1} y^l + 2x^{n-1} y^l$$

$$= \left(\frac{d-3}{y-1} + \frac{1}{x-2} + 2\right) x^{n-1} y^l = x^n y^l.$$

REMARK. Since $A(L,1) = 1$, the same proof shows that $A(L,n) \leqq x^{n-1} y^l$.

Now $A(n) \leqq x^n y^{n(d-3)} = (xy^{d-3})^n$; hence $\alpha(d) \leqq xy^{d-3}$. Using the previous remark and another simple argument based on [1, Lemma 1.4], it can be shown that $c_n(R) \leqq x^{n-1} y^{(n-d/2)(d-3)}$ for any algebra $R \in \mathscr{A}(d)$. This will not lead to a better estimation for $\alpha(d)$ since $\sup \sqrt[n]{x^{n-1} y^{(n-d/2)(d-3)}} = xy^{d-3}$.

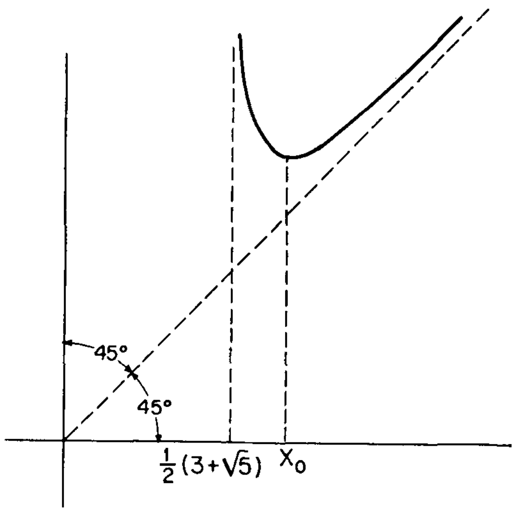
We use (3) to obtain values for x and y which minimize xy^{d-3} and this improves the previous estimation for $\alpha(d)$ which was $\alpha(d) < 3.4^{d-3}$. Our result is:

THEOREM. *For* $d \geqq 4$, $\alpha(d) < 3(d^2 - 7d + 16) = O(d^2)$.

PROOF. Let $u = d - 3$. Thus by (3), we have

$$xy^{d-3} = xy^u = x\left(1 + \frac{u(x-1)}{x^2 - 3x + 1}\right)^u$$

and we denote this function by $f(x)$. Since $y > 1$, it follows that $(u(x-1))/(x^2 - 3x + 1) > 0$ and since $x > 1$ we have also that $x > \frac{1}{2}(3 + \sqrt{5})$. Using elementary calculus, it can be shown that the graph of $f(x)$ for $x > \frac{1}{2}(3 + \sqrt{5})$ is of the following shape:



We have to locate x_0 for which $f(x)$ is minimum. It can be shown that $u^2 - u + 4 \leqq x_0 < u^2 - u + 5$ and that $f(u^2 - u + 4) < f(u^2 - u + 5)$. So we obtain $\alpha(d) \leqq f(u^2 - u + 4)$. Now $f(u^2 - u + 4) = (u^2 - u + 4)g(u)$ where

$$g(u) = \left(1 + \frac{u(u^2 - u + 3)}{(u^2 - u + 4)(u^2 - u + 1) + 1}\right)^u.$$

For $u \leqq 5$, one obtains after computations that $g(u) < 3$. For $u \geqq 6$, we have

$$\frac{u(u^2 - u + 3)}{(u^2 - u + 4)(u^2 - u + 1) - 1} < \frac{1}{u - 1}$$

and $g(u) < (1 + 1/(u-1))^u$. Since the sequence $(1 + 1/(u-1))^u \searrow e$ and for $u = 6$ it is already < 3, it follows that $g(u) < 3$ for all positive integral values of u; this implies that $f(u^2 - u + 4) < 3(u^2 - u + 4)$. Since $u = d - 3$, we get that $\alpha(d) < 3(d^2 - 7d + 16) = O(d^2)$. In fact, 3 can be replaced by $e + O(d^{-1})$.

4. The special cases $d = 4, 5$

We apply now the above methods to the special cases $d = 4$ and $d = 5$, and compare the results with those obtained from a computer. We wish to express our thanks to Mr. S. Libai for supplying the input, $A(L,n)$, to the C.D.C. 6600 computer of Tel Aviv University.

For $d = 4$, we have

$$f(x) = x\left(1 + \frac{x-1}{x^2 - 3x + 1}\right) \quad \text{and } x_0 = 4$$

(which is the value of $u^2 - u + 4$ for $u = 1$). Substituting this value, we get $f(4) = 6.4$ and so $\alpha(4) \leqq 6.4$.

For $d = 5$, we have

$$f(x) = x\left(1 + \frac{2(x-1)}{x^2 - 3x + 1}\right)^2, \quad 6 < x_0 < 7$$

and $f(x) < 13.92$ for $6 \leqq x \leqq 7$, so $\alpha(5) < 13.92$.

Turning now to the computer results, the recursive definition [1,4.1] was used for the computation of the system of numbers $A(L,n)$. The computations show, in both cases $d = 4$ and $d = 5$, that the ratio $A(n)/A(n-1)$ is increasing as far as it was computed, and since $A(1) = 1$ it follows that $\sqrt[n]{A(n)} \leqq A(n)/A(n-1)$. The ratio $A(n)/A(n-1)$ was computed for $n \leqq 50$ in the case $d=4$ and for $n \leqq 30$ in the case $d=5$. The last ten results in each case are given in the following table:

$d = 4$		$d = 5$	
n	$A(n)/A(n-1)$	n	$A(n)/A(n-1)$
41	6.165388	21	10.714756
42	6.170986	22	10.839524
43	6.176322	23	10.954959
44	6.181416	24	11.062068
45	6.186283	25	11.161718
46	6.190937	26	11.254660
47	6.195394	27	11.341548
48	6.199664	28	11.422953
49	6.203760	29	11.499378
50	6.207691	30	11.571266

This leads us to believe that, using (1) ,the result we have obtained for $d = 4,5$ (and probably for all $d \geqq 4$) cannot be significantly improved.

5. Application

By Corollary 5.2 [1], if $R \in \mathscr{A}(d)$ and $S \in \mathscr{A}(h)$ with $\{c_n(R)\}$, $\{c_n(S)\}$ the corresponding sequences of codimensions, then $R \otimes S$ will satisfy an identity of degree n if $c_n(R)\, c_n(S) < n!$ and hence if $(\alpha(d)\, \alpha(h))^n < n!$, i.e., $\alpha(d)\, \alpha(h) < \sqrt[n]{n!}$. Since $n/e < \sqrt[n]{n!}$, $\alpha(d) < 3(d^2 - 7d + 16)$, $\alpha(h) < 3(h^2 - 7h + 16)$, $R \otimes S$ will satisfy an identity of degree n, if $n > 9e(d^2 - 7d + 16)(h^2 - 7h + 16)$. This result may be formulated also as follows: $\mathscr{A}(d) \otimes \mathscr{A}(h) \subseteq \mathscr{A}(n)$ if n satifies the previous condition. In fact $9 = 3^2$ can be replaced by $e^2 + O(k^{-1})$, $k = \min\{d, h\}$.

6. The codimension power series

With a PI-algebra R, we attach a power series $f_R(t) = \sum_{n=1}^{\infty} c_n t^n$, where $c_n = c_n(R)$ for $n \geqq 1$. If $R \in \mathscr{A}(d)$, then $c_n(R) \leqq \alpha(d)^n$; hence the series $f_R(t)$ is convergent at least for $|t| < \alpha(d)^{-1}$. Note that $f_R(t)$ can be defined for any algebra R. But if R is not a PI-algebra, then $c_n(R) = n!$ and so the class of PI-algebras is the class of those algebras for which $f_R(t)$ is convergent for some $t \neq 0$.

The class of nilpotent algebras may be characterized as the class of those algebras for which $f_R(t)$ is a polynomial. Commutative algebras R are characterized by $c_n(R) \leqq 1$ and if R is commutative and not nilpotent, then $c_n(R) = 1$, and so in this case, $f_R(t)$ is given precisely. Other examples of PI-algebras for which $f_R(t)$ can be computed will follow from the following result which can be derived easily using the K-modules $V_n(x)$ and $V_n^{(h)}(x)$ defined in [1].

Let $Q \neq \{0\}$ be a T-ideal in the free algebra $K[X]$ (without 1) and let $R = K[X]/Q$. Choose $z \in \{x\}$ and let $S = K[X]/T(zQ)$, where $T(zQ)$ is the T-ideal generated by zQ in $K[X]$. Then $c_n(S) = nc_{n-1}(R)$.

In particular if Q is the T-ideal generated by $[x_1, x_2]$ and $P = T(z[x_1, x_2])$, then $c_n(K[X]/P) = n$. Using this result successively we get that if $z_1, \cdots, z_m \in \{x\}$ and $P_m = T(z_1, \cdots, z_m[x_1, x_2])$, then $c_n(K[X]/P_m) = n(n-1)\cdots(n-m+1)$ for $n \geqq m$ and $c_n = n!$ for $n < m$.

Reference

1. A. Regev, *Existence of identities in $A \otimes B$*, Israel J. Math. **11** (1972), 131–152.

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